MODELING OF EIGENVALUES AND EIGENFUNCTIONS OF HEAT- AND MASS-TRANSFER SYSTEMS

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We discuss a method of modeling the eigenvalues and eigenfunctions of heat- and mass-transfer systems on analog computers. The bases for the modeling method are given, and an example is presented.

Finding the eigenvalues and eigenfunctions of heat- and mass-transfer systems in analytic form is very difficult even in the simplest problems. Knowledge of them frequently enables one to draw far reaching qualitative conclusions about the phenomena described by the corresponding equations. In particular, a knowledge of even the first few eigenvalues and eigenfunctions of heat- and mass-transfer systems permits the use of the theory and conclusions of the so-called regular thermal conditions [1, 2].

We discuss a method for finding approximate eigenvalues and eigenfunctions of heat- and mass-transfer systems by modeling the appropriate Sturm–Liouville equations on analog computers. The methods used here are described in [3, 4].

Suppose the heat- and mass-transfer equations in matrix form are

$$\frac{\partial U}{\partial \tau} = \frac{\partial}{\partial x}\left(a\frac{\partial U}{\partial x}\right) - bU;\ l_1 < x < l_2 \tag{1}$$

with the initial conditions

$$U(0,\ x) = \varphi(x) \tag{2}$$

and the homogeneous boundary conditions

$$[\alpha^{(1)}U_x - \beta^{(1)}U]_{x=l_1} = 0,\ [\alpha^{(2)}U_x + \beta^{(2)}U]_{x=l_2} = 0. \tag{3}$$

Here $a = (a_{ij})$, $\beta^{(k)} = (\beta^{(k)}_{ij})$, $\alpha^{(k)} = (\alpha^{(k)}_{ij})$, $b = (b_{ij})$, ($\alpha^{(k)}_{12} = \alpha^{(k)}_{21} \equiv 0$, $\det a \neq 0$; i, j, k = 1, 2) are given square matrices whose elements depend on x, and $U = \begin{vmatrix} U_1 \\ U_2 \end{vmatrix}$, $\varphi = \begin{vmatrix} \varphi^1 \\ \varphi_2 \end{vmatrix}$ are column matrices. It is required to find the eigenvalues and eigenfunctions of system (1)-(3).

It is more convenient to write (1) in the form

$$\frac{\partial U}{\partial \tau} = a\frac{\partial^2 U}{\partial x^2} + a'\frac{\partial U}{\partial x} - bU, \tag{4}$$

where a' is the derivative of matrix a with respect to x. Setting

$$U = \begin{bmatrix} U_1(x,\ \tau) \\ U_2(x,\ \tau) \end{bmatrix} = T(\tau)X(x) = T(\tau)\begin{bmatrix} X_1(x) \\ X_2(x) \end{bmatrix}$$

and separating variables in (4) and (3) we must find nontrivial solutions $X^{(k)}(x)$ — the eigenfunctions — of the following Sturm–Liouville problem:

$$X'' + a^{-1}a'X' + a^{-1}(\lambda^2 E - b)X = 0, \tag{5}$$

$$[\alpha^{(1)}X' - \beta^{(1)}X]_{x=l_1} = 0,\ [\alpha^{(2)}X' + \beta^{(2)}X]_{x=l_2} = 0, \tag{6}$$

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where a^{-1} is the inverse of matrix a, and E is the second-order unit matrix. We denote the elements of the matrix $a^{-1}a'$ by $-\gamma_{ij}$, and the elements of $a^{-1}(\lambda^2 E - b)$ by $-\delta_{ij}$. Then (5) takes the form

$$X_1'' - \gamma_{11}X_1' - \gamma_{12}X_2' - \delta_{11}X_1 - \delta_{12}X_2 = 0, \quad X_2'' - \gamma_{21}X_1' - \gamma_{22}X_2' - \delta_{21}X_1 - \delta_{22}X_2 = 0. \tag{7}$$

We set $X_1 = V_1$, $X_1' = V_2$, $X_2 = V_3$, and $X_2' = V_4$. Then we obtain from (7) and (6) the following boundary-value problem for determining the eigenvalues and eigenfunctions:

$$\frac{dV_1}{dx} = V_2, \quad \frac{dV_2}{dx} = \delta_{11}V_1 + \gamma_{11}V_2 + \delta_{12}V_3 + \gamma_{12}V_4, \quad \frac{dV_3}{dx} = V_4, \quad \frac{dV_4}{dx} = \delta_{21}V_1 + \gamma_{21}V_2 + \delta_{22}V_3 + \gamma_{22}V_4 \tag{8}$$

with the boundary conditions

$$[\alpha_{11}^{(1)}V_2 - \beta_{11}^{(1)}V_1 - \beta_{12}^{(1)}V_3]_{x=l_1} = 0, \quad [\alpha_{22}^{(1)}V_4 - \beta_{21}^{(1)}V_1 - \beta_{22}^{(1)}V_3]_{x=l_1} = 0, \quad [\alpha_{11}^{(2)}V_2 + \beta_{11}^{(2)}V_1 + \beta_{12}^{(2)}V_3]_{x=l_2} = 0, \quad [\alpha_{22}^{(2)}V_4 + \beta_{21}^{(2)}V_1 + \beta_{22}^{(2)}V_3]_{x=l_2} = 0. \tag{9}$$

We seek a nontrivial solution of problem (8) and (9) in the form

$$V_k = \sum_{j=1}^{4} c_j V_{kj} \ (k = 1, \ldots, 4), \tag{10}$$

where (V_{kj}) is the matrix of the fundamental solutions of Eqs. (8), and the c_j are unknown constants.

The functions V_{kj} can be found, for example, as solutions of the following Cauchy problems for (8):

$$V_{k1}(l_1) = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases}; \quad V_{k2}(l_1) = \begin{cases} 1, & k = 2 \\ 0, & k \neq 2 \end{cases}; \quad V_{k3}(l_1) = \begin{cases} 1, & k = 3 \\ 0, & k \neq 3 \end{cases}; \quad V_{k4} = \begin{cases} 1, & k = 4 \\ 0, & k \neq 4 \end{cases}. \tag{11}$$

Substituting V_k from (10) into boundary conditions (9), we obtain, on the one hand, a system of equations for determining the unknowns c_j, and, on the other hand, by equating the determinant of this system to zero, we obtain a certain condition (Λ) which must be satisfied when λ is equal to an eigenvalue λ_k.

Thus, for the first boundary-value problem in (8) and (9)

$$c_1 = c_3 = 0, \ c_4 = 1, \ c_2 = -V_{14}(l_2)/V_{12}(l_2).$$

Condition (Λ) in this case takes the form

$$V_{12}(l_2)V_{34}(l_2) - V_{14}(l_2)V_{32}(l_2) = 0.$$

For the second boundary-value problem in (8) and (9)

$$c_2 = c_4 = 0, \ c_3 = 1, \ c_1 = -V_{24}(l_2)/V_{21}(l_2)$$

and condition (Λ) has the form

$$V_{21}(l_2)V_{34}(l_2) - V_{31}(l_2)V_{24}(l_2) = 0.$$

The matrix of the fundamental solutions (V_{kj}), the eigenvalues λ_k, and the eigenfunctions $X^{(k)}$ are conveniently found on analog computers. With a fixed λ we solve Eqs. (8) four times on the computer with initial conditions (11) and check to see if condition (Λ) is satisfied. Varying λ, i.e., varying the coefficients δ_{ij} in the block diagram of the model, we again solve Eqs. (8) with conditions (11), trying to satisfy condition (Λ). The eigenvalues λ_k of system (1) are found in this way. The values found for λ_k make it possible to determine the initial values

$$\{V_1(l_1), V_2(l_1), V_3(l_1), V_4(l_1)\}_{\lambda=\lambda_k}. \tag{12}$$

Then by solving system (8) with the initial conditions (12) we find the eigenfunctions $X^{(k)}(x)$.

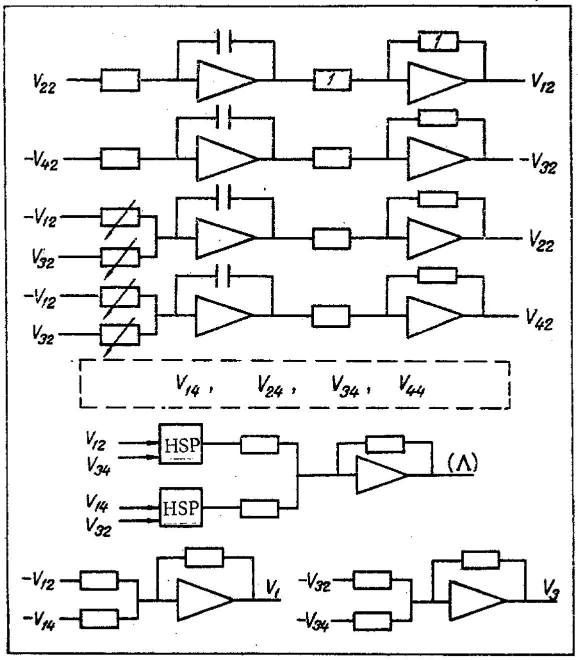

Fig. 1. Block diagram of the model of problem (13), expression (10), and condition (Λ).

It should be noted that the whole modeling procedure described is performed for a fixed block diagram determined by Eqs. (8); only the initial conditions and the coefficients δ_{ij} which depend on λ are varied.

As an illustration of the method we present the results of modeling the following problems on an MN-14 analog computer: Find the eigenvalues and eigenfunctions of a heat- and mass-transfer system described by the dimensionless equations

$$\frac{\partial U_1}{\partial \tau} = 10\frac{\partial^2 U_1}{\partial x^2} + 5\frac{\partial^2 U_2}{\partial x^2}, \quad 0 < x < 1, \quad \frac{\partial U_2}{\partial \tau} = 0.1\frac{\partial^2 U_1}{\partial x^2} + 0.2\frac{\partial^2 U_2}{\partial x^2} \tag{13}$$

with the initial conditions

$$U_1(0, x) = \varphi_1(x), \; U_2(0, x) = \varphi_2(x).$$

System (8) takes the form

$$\frac{dV_1}{dx} = V_2,$$

$$\frac{dV_2}{dx} = -\frac{\lambda^2}{1.5}(0.2V_1 - 5V_3),$$

$$\frac{dV_3}{dx} = V_4,$$

$$\frac{dV_4}{dx} = -\frac{\lambda^2}{1.5}(0.1V_1 - 10V_3)$$

with the boundary conditions $V_1(0) = V_3(0) = V_1(1) = V_3(1) = 0$. Here

$$c_1 = c_3 = 0, \; c_4 = 1, \; c_2 = -V_{14}(1)/V_{12}(1),$$

and condition (Λ) takes the form

$$V_{12}(1)V_{34}(1) - V_{14}(1)V_{32}(1) = 0.$$

The block diagram of the model corresponding to Eqs. (13), condition (Λ), and the expressions $V_k = \sum_{j=1}^{4} c_j V_{kj}$

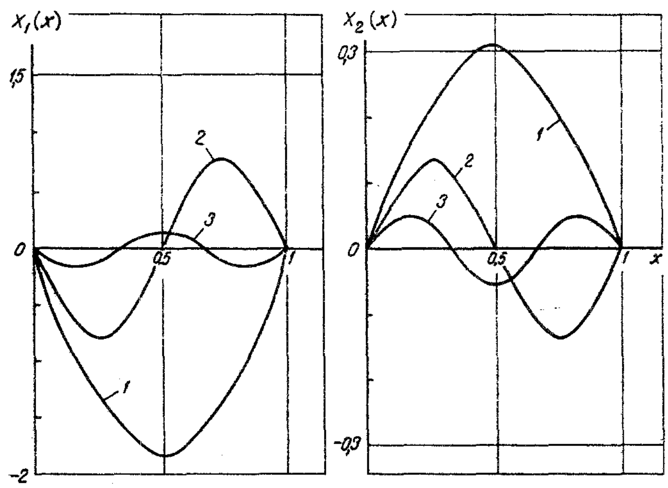

Fig. 2. Eigenfunctions of problem (13) corresponding to the eigenvalues λ_1, λ_2, and λ_3.

TABLE 1. Modeling of Eigenvalues and Eigenfunctions of Heat- and Mass-Transfer Systems

No.	λ_{mach}	No.	λ_{mach}
1	1,187	6	7,580
2	2,387	7	8,832
3	3,715	8	10,100
4	4,975	9	11,220
5	6,269	10	12,210

is shown in Fig. 1. The arrows denote resistances for modeling the coefficients which depend on λ. In the example, the modeling process is shortened as a result of the following considerations:

1. Since $c_1 = c_3 = 0$, the functions V_{k1} and V_{k3} are not zero, and, therefore, it is sufficient to model Eqs. (8) twice to find V_{k2} and V_{k4}.

2. The block diagram of Fig. 1 was constructed to obtain V_{k2} and V_{k4}, test condition (Λ), and find the functions V_1 and V_3 simultaneously.

The scales of the variables and the calculation of the coefficients are too obvious to present.

Figure 2 shows graphs of the first three eigenfunctions of this problem corresponding to the eigenvalues λ_1, λ_2, and λ_3; and Table 1 lists the first ten eignevalues found by modeling.

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